Answer the Following Questions:

(1) Show that the function $w = z^2 + e^{iz}$ satisfy Cauchy Remman's equations.

(2) Prove that $\overline{\sin z} = \sin \bar{z}$.

(3) Find $\int (z^2 + 3z)dz$ on the line $y = x$ from the point(0,0) to (3,3)

(4) Evaluate $\int_C \frac{z^2+5}{z} dz$ where C is the circle $z = 2e^{i\theta}$.

Solution:

Answer (1)

$$w = z^2 + e^{iz} = (x+iy)^2 + e^{i(x+iy)} = x^2 - y^2 + 2ixy + e^{-y+ix}$$

$$= x^2 - y^2 + 2ixy + e^{-y}(\cos x + i\sin x)$$

$$= \left(x^2 - y^2 + e^{-y}\cos x\right) + i\left(2xy + e^{-y}\sin x\right)$$

$$u(x,y) = \left(x^2 - y^2 + e^{-y}\cos x\right)$$

$$v(x,y) = \left(2xy + e^{-y}\sin x\right)$$

$$\frac{\partial u}{\partial x} = 2x - e^{-y}\sin x \qquad \frac{\partial v}{\partial x} = 2y + e^{-y}\cos x$$

$$\frac{\partial u}{\partial y} = -2y - e^{-y}\cos x \qquad \frac{\partial v}{\partial y} = 2x - e^{-y}\sin x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The function satisfy Cauchy Remman's equations

Answer (2)

$$\overline{\sin z} = \overline{\sin(x+iy)}$$

$$= \overline{\sin x\cos(iy) + \cos x\sin(iy)} = \overline{\sin x\cosh y + i\cos x\sinh y}$$

$$= \sin x\cosh y - i\cos x\sinh y = \sin x\cos(iy) - \cos x\sin(iy)$$

$$= \sin(x-iy) = \sin\bar{z} \qquad \text{then } \overline{\sin z} = \sin\bar{z}$$

Answer (3)

$z = x + iy$ and the path of integration is $y = x$ substitute in z we find

$z = x + ix = (1+i)x$ and differentiate then $dz = (1+i)dx$

$$\int (z^2 + 3z)dz = \int_0^3 \left[(1+i)^2 x^2 + 3(1+i)x\right](1+i)dx$$

$$= \int_0^3 \left[(1+i)^3 x^2 + 3(1+i)^2 x\right]dx$$

$$= \left[(1+i)^3 \frac{x^3}{3} + 3(1+i)^2 \frac{x^2}{2}\right]_0^3 = \boxed{9(1+i)^3 + \frac{27}{2}(1+i)^2}$$

you can simplify the answer as following

$$= 9(1+3i+3i^2+i^3) + \frac{27}{2}(1+2i+i^2) = 9(1+3i-3-i) + \frac{27}{2}(1+2i-1)$$

$$= 9(2i-2) + 27i = \boxed{-18+45i}$$

Answer (4)

$\int_C \frac{z^2+5}{z}dz$ where C is the circle $z = 2e^{i\theta}$.

$z = 2e^{i\theta}$ then $dz = 2ie^{i\theta}d\theta$ then the integral

$$\int_C \frac{z^2+5}{z}dz = \int_0^{2\pi} \left(\frac{4e^{2i\theta}+5}{2e^{i\theta}}\right) 2ie^{i\theta}d\theta$$

$$= \int_0^{2\pi} i\left(4e^{2i\theta}+5\right)d\theta = i\left(\frac{4e^{2i\theta}}{2i} + 5\theta\right)_0^{2\pi} = i\left(-2ie^{2i\theta}+5\theta\right)_0^{2\pi}$$

$$= i\left(-2ie^{4\pi i} + 2ie^0 + 5(2\pi)\right)_0^{2\pi} = \boxed{10\pi i}$$

Another solution

Applying Cauchy Theorem

The circle $z = 2e^{i\theta}$ contains the point $z = 0$ and compare with the theorem

$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$ we find that $a = 0, f(z) = z^2 + 5$ then $f(a) = f(0) = 5$

$$\int_C \frac{z^2+5}{z}dz = 2\pi i(5) = 10\pi i$$
